Detailed numerical investigation of 90 scattering of energetic particles interacting with magnetic turbulence

G. Qin and A. Shalchi

Citation: Physics of Plasmas (1994-present) 21, 042906 (2014); doi: 10.1063/1.4873895
View online: http://dx.doi.org/10.1063/1.4873895
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/21/4?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in
Cosmic ray diffusion: Detailed investigation of a recent model
Phys. Plasmas 18, 082305 (2011); 10.1063/1.3625277

Anomalous, non-Gaussian transport of charged particles in anisotropic magnetic turbulence
Phys. Plasmas 14, 012311 (2007); 10.1063/1.2434795

Gyrotropic guiding-center fluid theory for turbulent inhomogeneous magnetized plasma
Phys. Plasmas 13, 072903 (2006); 10.1063/1.2220006

Dynamics of turbulence spreading in magnetically confined plasmas
Phys. Plasmas 12, 032303 (2005); 10.1063/1.1853385

Particle diffusion in random fields: Time-nonlocal description and numerical simulations
Phys. Plasmas 10, 58 (2003); 10.1063/1.1525017
Detailed numerical investigation of $90^\circ$ scattering of energetic particles interacting with magnetic turbulence

G. Qin$^1$ and A. Shalchi$^2$

$^1$State Key Laboratory of Space Weather, National Space Science Center, Chinese Academy of Sciences, Beijing 100190, China
$^2$Department of Physics and Astronomy, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada

(Received 25 February 2014; accepted 17 April 2014; published online 28 April 2014)

In the present paper, we re-visit a well-known problem in diffusion theory, namely the $90^\circ$ scattering problem. We use a test-particle code to compute the pitch-angle Fokker-Planck coefficient at $90^\circ$ for different values of the turbulent magnetic field strength and the magnetic rigidity. We consider a slab model and compare our numerical findings with the analytical result provided by second-order quasilinear theory. We show that the latter theory accurately describes $90^\circ$ scattering. We also replace the slab model by a more realistic two-component model to explore the influence of the turbulence model on $90^\circ$ scattering. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4873895]

I. INTRODUCTION

The physical environment considered in the current article is well-known in space and astrophysics and can also be found in fusion devices. We assume a superposition of an ordered magnetic field (also called mean field or background field) $B_0$ and a turbulent component $\delta B$. Energetic particles which are electrically charged (e.g., electrons, positrons, or protons) interact with this field configuration and experience scattering. This effect is described by diffusion theory. Analytical and numerical results for the motion of energetic particles in such scenarios are important in order to understand cosmic ray propagation and acceleration.

A. Pitch-angle scattering and the $90^\circ$ problem

A fundamental parameter in order to describe the propagation of a particle through a turbulent plasma is the diffusion coefficient along the mean magnetic field. In the following, we will call this parameter the parallel diffusion coefficient $\kappa_\parallel$ as it is usually done in the literature. This parameter enters the cosmic ray transport equation$^4$ and is, therefore, important to understand the transport but also the acceleration of energetic particles such as cosmic rays. Such transport equations are basically extended diffusion equations.

A more fundamental description of the transport can be achieved by using a Fokker-Planck equation in which the particles are described in a six-dimensional phase-space. In this case, the characteristic quantity is the so-called pitch-angle Fokker-Planck coefficient $D_{\mu\mu}(\mu)$, where we have used the pitch-angle cosine $\mu = v_\parallel/v$ (here, $v$ is the particle speed and $v_\parallel$ is the velocity component along the mean magnetic field). It is also well-known that the two parameters $D_{\mu\mu}(\mu)$ and $\kappa_\parallel$ are related to each other via the relation$^2$

$$\kappa_\parallel = \frac{v^2}{8} \int_{-1}^{+1} d\mu \left( \frac{1 - \mu^2}{D_{\mu\mu}(\mu)} \right).$$  

(1)

In order to calculate the parallel spatial diffusion coefficient, one has to derive an analytical form for the pitch-angle Fokker-Planck coefficient first and then one has to solve the integral in Eq. (1).

The standard tool to calculate the parameter $D_{\mu\mu}(\mu)$ is the quasilinear theory (QLT) developed by Jokipii.$^3$ Within this approach, one replaces real particle orbits by unperturbed particle trajectories if a transport parameter is calculated. Thus, QLT can be seen as a first-order perturbation theory. A few years after the quasilinear approach had been established, it was noticed that the theory cannot describe $90^\circ$ scattering correctly. In some cases, such as for isotropic turbulence, this is even leading to a singularity in the integral of Eq. (1). The inability of QLT for describing scattering at such pitch-angles is usually called the $90^\circ$ problem. Several nonlinear theories have been developed in the 1970s (see, e.g., Refs. 4–9) to solve this problem. A very promising approach is the second-order quasilinear theory derived in Ref. 10 which is discussed in Sec. 1B.

B. The second-order quasilinear theory

An extension of QLT was developed in Shalchi.$^{10}$ The latter theory, which is called the second-order quasilinear theory (SOQLT), uses perturbed orbits to compute the parameter $D_{\mu\mu}(\mu)$ (see Refs. 10 and 11 for details). SOQLT provides not just a finite value for $D_{\mu\mu}(\mu = 0)$, it also predicts that scattering at $90^\circ$ is usually very strong. The theory discussed here was able to accurately describe parallel diffusion in isotropic turbulence.$^{12}$ Furthermore, SOQLT can also be used for a spectrum for which the inertial range spectral index is larger than $s = 5/3$ which is the value suggested by Kolmogorov’s theory of turbulence.$^{13}$ For such steep spectra, the theory agrees very well with performed simulations.$^{14}$

For slab turbulence and for a standard spectrum (see below), SOQLT provides the following form for the pitch-angle Fokker-Planck coefficient at $90^\circ$

$$D_{\mu\mu}(\mu = 0) \equiv D_{\mu\mu}(\mu = 0) \frac{l_{\text{lab}}}{v} \approx \sqrt{\pi} \Gamma(s/2) C(s) R s^{-2} \left( \frac{\delta B}{B_0} \right)^{s+1}.$$  

(2)
Here, we have used the (normalization) function

\[ C(s) = \frac{\Gamma(s/2)}{2\sqrt{\pi}\Gamma((s-1)/2)}. \]  

(3)

The parameter \( s \) is the inertial range spectral index, \( R = R_l/l_{slab} \) is the Larmor radius divided by the slab bend-over scale \( l_{slab} \). \( \delta B \) is the turbulent magnetic field, and \( B_0 \) is the mean field. Furthermore, we have used the Gamma-function \( \Gamma(z) \).

It also has to be emphasized that formula (2) is only correct if the restriction \( R\delta B/B_0 \ll 1 \) holds. In the present paper, we perform test-particle simulations to compute numerically the pitch-angle Fokker-Planck coefficient at \( \mu = 0 \) corresponding to pitch-angles of 90°. It is our intention to check the validity of Eq. (2) and to explore how scattering at 90° becomes different if the slab model of turbulence is replaced by a two-component model.

II. TEST-PARTICLE SIMULATIONS

In our simulations, we have to choose a certain turbulence model. In the following, we use the two-component model. The aforementioned slab model is a special case of this model.

A. The two-component model

A prominent model for solar wind turbulence is the so-called two-component model. The latter model can be confirmed by using extensive analyses of solar wind data. According to such observations, magnetic correlations in the solar wind have the form of a so-called maltese cross. Similar measurements were done in the following years which have confirmed this structure of interplanetary turbulence (for a review see Ref. 22). The model discussed here is also supported by numerical simulations and analytical work.

In the two-component model, the magnetic correlation tensor in the \( \vec{k} \)-space has the form

\[ P_{lm}(\vec{k}) = P_{lm}^{slab}(\vec{k}) + P_{lm}^{2D}(\vec{k}). \]

(4)

The latter quantity is defined as \( P_{lm}(\vec{k}) = \langle \delta B_l(\vec{k})\delta B_m(\vec{k}) \rangle \), where \( \langle \ldots \rangle \) denotes the ensemble average. The tensor of the slab modes has the form

\[ P_{lm}^{slab}(\vec{k}) \]

(5)

with \( l, m = x, y \). Here, we have used the Kronecker delta and the Dirac delta, respectively. The other tensor components are zero due to the solenoidal constraint. The two-dimensional modes are defined by

\[ P_{lm}^{2D}(\vec{k}) = \frac{\delta^{slab}(\vec{k})}{\delta_{lm} - k_{lx}k_{ly}}. \]

(6)

if \( l, m = x, y \) and \( P_{zz} = P_{zm} = P_{zx} = 0 \). In this particular model, the magnetic field vector as well as the spatial dependence are two-dimensional. Above we have used the spectrum of the slab modes \( g^{slab}(k_l) \) and the spectrum of the two-dimensional modes \( g^{2D}(k_\perp) \) which are discussed in the following.

B. The model spectra

For the two spectra introduced above, we use the following forms:

\[ g^{slab}(k_l) = \frac{C(s)}{2\pi} \frac{\delta B_{slab}^2 l_{slab}}{1 + (k_l l_{slab})^2} \]

(7)

and

\[ g^{2D}(k_\perp) = \frac{2D(s,q)}{\pi} \frac{\delta B_{2D}^2 l_{2D}}{1 + (k_\perp l_{2D})^2} \]

(8)

In the latter two formulas, we used the slab bend-over scale \( l_{slab} \), the two-dimensional bendover scale \( l_{2D} \), the magnetic field strength of the slab modes \( \delta B_{slab} \), and the magnetic field strength of the two-dimensional modes \( \delta B_{2D} \), respectively. Furthermore, we have used the inertial range spectral index \( s \). The first model spectrum is in agreement with the one used by Bieber et al. The model for the two-dimensional spectrum has been proposed by Shalchi and Weinhorst. For the slab modes, we employ a spectrum which is perfectly flat at large scales and for the two-dimensional modes we allow a general spectrum in the energy range which is controlled by the parameter \( q \). The latter parameter is usually called the energy range spectral index. The physical consequences of the different forms of the spectrum are discussed in Ref. 30. In Eqs. (7) and (8), we have also used the normalization functions \( D(s,q) = \Gamma((s+q)/2)/(2\Gamma((s+1)/2)\Gamma((q+1)/2)) \) and \( C(s) \equiv D(s,q = 0) \), where the latter function was already defined in Eq. (3). Here, we used again the gamma-function \( \Gamma(z) \) and the two spectra are correctly normalized for \( s > 1 \) and \( q > -1 \).

C. Our code

Here, we present a comparison between the theoretical results discussed previously with numerical simulations. The simulation code used in this paper to calculate charged particle transport parameters has been used before (see Refs. 31–34 for details). Test-particle simulations have also been performed by other authors.

In our simulations, we use the turbulence model described in Secs. II A and II B to compute the scattering parameter \( D_{\mu\nu} \) as it was done in Ref. 39. In the latter paper, the pitch-angle Fokker-Planck coefficient has been calculated for specific turbulence and particle parameters. In the current paper, however, we focus on 90° scattering and we explore how \( D_{\mu\nu} \) depends on these parameters.

In our simulations, we solve the Newton-Lorentz equation based on a fourth-order Runge-Kutta method with adaptive step size control. More details about the used numerical method can be found in the aforementioned papers. The slab spectrum is created with a periodic box of size...
This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions.

10,000/l_{slab} and N_z = 4,194,304 points and the two-dimensional spectrum is created with a periodic box of size 100/l_{slab} \times 100/l_{slab} and N_x \times N_y = 4096 \times 4096 points. The perpendicular correlation scale for 2D turbulence, l_{2D} is set to be 1/10 of the parallel bendover scale. To study the pitch-angle diffusion coefficients, we calculate 2000 test-particle trajectories for each model turbulence configuration. From the trajectories, we calculate running pitch-angle diffusion coefficients D_{\mu\mu}(\mu, t).

III. RESULTS

We perform simulations for different turbulence models and parameters. We also compare the simulations with each other and analytical results.

A. Analytical forms

The main purpose of the current article is to explore how the pitch-angle Fokker-Planck coefficient D_{\mu\mu}(\mu = 0) depends on the parameters R = R/l_{slab} and \delta B/B_0. In the following, we assume that

\[ D_{\mu\mu}(\mu = 0) \sim \left(\frac{\delta B}{B_0}\right)^{\alpha} R^\beta. \]  

(9)

Below, we simulate particle transport for slab turbulence and two-component turbulence. In the former case, the prediction of SOQLT (see Eq. (2)) is \( \alpha = s + 1 \) and \( \beta = s - 2 \).

B. Simulations for slab turbulence

Here, we consider pure slab turbulence. We perform the simulations for three different values of the inertial range

<table>
<thead>
<tr>
<th>Spectral index</th>
<th>Simulations for slab</th>
<th>SOQLT</th>
<th>Simulations for slab/2D</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>( \alpha )</td>
<td>( \beta )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>5/3</td>
<td>2.11</td>
<td>-0.269</td>
<td>2.67</td>
</tr>
<tr>
<td>2</td>
<td>3.01</td>
<td>-0.0011</td>
<td>3</td>
</tr>
<tr>
<td>2.4</td>
<td>3.14</td>
<td>0.548</td>
<td>3.4</td>
</tr>
</tbody>
</table>

spectral index, namely \( s = 5/3, s = 2, \) and \( s = 2.4 \). We look at different values for \( R \) and \( \delta B/B_0 \) to obtain the parameters \( \alpha \) and \( \beta \). In Fig. 1, we show the pitch-angle Fokker-Planck coefficient versus the time as an example. As shown there, we obtain indeed a diffusive pitch-angle scattering regime. All results obtained from the simulations are summarized in Table I.

Figs. 2 and 3 show the scattering parameter D_{\mu\mu}(\mu = 0) versus \( \delta B/B_0 \) and \( R \), respectively. For the simulations shown there, we set the inertial range spectral index \( s = 5/3 \) corresponding to the value proposed by Kolmogorov.\(^{13}\) According to our fits, we find \( \alpha = 2.11 \) and \( \beta = -0.269 \). These values are very close to the prediction made by SOQLT, where we have \( \alpha = 2.7 \) and \( \beta = -0.33 \).

Figs. 4 and 5 show the scattering parameter D_{\mu\mu}(\mu = 0) versus \( \delta B/B_0 \) and \( R \) as before but now we set \( s = 2.4 \). For this value of the inertial range spectral index, quasinlinear theory is problematic but not SOQLT. According to our fits, we now find \( \alpha = 3.01 \) and \( \beta = -0.0011 \). These values agree perfectly with SOQLT which predicts \( \alpha = 3 \) and \( \beta = 0 \).

Figs. 6 and 7 show the scattering parameter D_{\mu\mu}(\mu = 0) versus \( \delta B/B_0 \) and \( R \) as before but now we set \( s = 2.4 \) corresponding to a steep spectrum. According to our fits, we now find \( \alpha = 3.14 \) and \( \beta = 0.548 \). These values agree well with SOQLT which predicts \( \alpha = 3.40 \) and \( \beta = 0.4 \).
Above we have considered three different values for the inertial range spectral index $s$ and we have employed the slab model. For all cases, we ran the simulations and we found values for the parameters $a$ and $b$. In all cases, our simulations agree well with the predictions made by SOQLT. It seems that SOQLT can describe pitch-angle scattering at $90^\circ$ with high accuracy. The results are summarized in Table I.

C. Simulations for two-component turbulence

In the current section, we replace the slab model used in the previous paragraph by the two-component model. In agreement with the solar wind observations described by Bieber et al.\textsuperscript{41} we assume that the slab contribution to the total magnetic energy is 20\% and the contribution of the two-dimensional modes is 80\%. Therefore, we set $\delta B^2_{\text{slab}}/\delta B^2 = 0.2$ and $\delta B^2_{2D}/\delta B^2 = 0.8$. For the slab modes, we employ the same spectrum as above and for the two-dimensional modes we set $q = 1.5$ in agreement with Matthaeus et al.\textsuperscript{30}

As above, we perform the simulations for three different values of the inertial range spectral index, namely $s = 5/3$, $s = 2$, and $s = 2.4$. Again we look at different values for $R$ and $\delta B/B_0$. In the following, we assume that the form (9) is still valid for two-component turbulence and we determine the parameters $a$ and $\beta$ as it was done for the slab model. We like to note that there is not necessarily a simple power-law dependence as assumed in Eq. (9). In the current paragraph, however, we fit such power-law dependences on the simulations and we find good agreement.

Figs. 8 and 9 show the scattering parameter $D_{\mu\beta}(\mu = 0)$ versus $\delta B/B_0$ and $R$, respectively. For the simulations shown...
there, we set the inertial range spectral index $s = 5/3$ corresponding to value proposed by Kolmogorov.$^{13}$

According to our fits, we find $\alpha = 1.78$ and $\beta = -0.556$ which is slightly different compared to the values we obtained numerically for pure slab turbulence ($\alpha = 2.11$ and $\beta = -0.269$).

Figs. 10 and 11 show the scattering parameter $D_{\mu\nu}(\mu = 0)$ versus $\delta B/B_0$ and $R$ as before but now we set $s = 2$. According to our fits, we now find $\alpha = 1.98$ and

$\beta = -0.113$. This is very different compared to the slab results where we found $\alpha = 3.01$ and $\beta = -0.0011$.

Figs. 12 and 13 show the scattering parameter $D_{\mu\nu}(\mu = 0)$ versus $\delta B/B_0$ and $R$ as before but now we set $s = 2.4$. According to our fits, we now find $\alpha = 2.00$ and $\beta = 0.23$. Again, this is very different compared to the slab results where we found $\alpha = 3.14$ and $\beta = 0.548$.

In Figs. 8–13, we have explored pitch-angle scattering at 90° for a two-component turbulence model. We have shown that the results are different compared to the values we have obtained for slab turbulence. We like to emphasize that SOQLT was formulated for slab turbulence and, therefore, we cannot compare this theory with the findings of the current paragraph. The results are summarized in Table I.

**IV. SUMMARY AND CONCLUSION**

The 90° scattering problem is well-known in diffusion theory. Progress has been achieved due to the derivation of the second-order quasilinear theory (SOQLT).$^{10}$ The latter theory was developed for slab turbulence. In the current article, we performed test-particle simulations to compute the pitch-angle Fokker-Planck coefficient $D_{\mu\nu}$ at $\mu = 0$ corresponding to 90°. We explored how $D_{\mu\nu}(\mu = 0)$ depends on the parameter $R = R_{\nu}/l_{\text{lab}}$ and the magnetic field ratio $\delta B/B_0$. 

FIG. 9. The pitch-angle Fokker-Planck coefficient $D_{\mu\nu}(\mu = 0)$ versus $R = R_{\nu}/l_{\text{lab}}$ for $\delta B/B_0 = 0.1$, $s = 5/3$, and two-component turbulence. We have shown the test-particle simulations (dots) versus a fit (dotted line). The fit indicates that $\beta = -0.556$.

FIG. 10. The pitch-angle Fokker-Planck coefficient $D_{\mu\nu}(\mu = 0)$ versus the ratio $\delta B/B_0$ for $R = R_{\nu}/l_{\text{lab}} = 0.05$, $s = 2$, and two-component turbulence. We have shown the test-particle simulations (dots) versus a fit (dotted line). The fit indicates that $\alpha = 1.98$.

FIG. 11. The pitch-angle Fokker-Planck coefficient $D_{\mu\nu}(\mu = 0)$ versus $R = R_{\nu}/l_{\text{lab}}$ for $\delta B/B_0 = 0.1$, $s = 2$, and two-component turbulence. We have shown the test-particle simulations (dots) versus a fit (dotted line). The fit indicates that $\beta = -0.113$.

FIG. 12. The pitch-angle Fokker-Planck coefficient $D_{\mu\nu}(\mu = 0)$ versus the ratio $\delta B/B_0$ for $R = R_{\nu}/l_{\text{lab}} = 0.05$, $s = 2.4$, and two-component turbulence. We have shown the test-particle simulations (dots) versus a fit (dotted line). The fit indicates that $\alpha = 1.98$.

FIG. 13. The pitch-angle Fokker-Planck coefficient $D_{\mu\nu}(\mu = 0)$ versus $R = R_{\nu}/l_{\text{lab}}$ for $\delta B/B_0 = 0.1$, $s = 2.4$, and two-component turbulence. We have shown the test-particle simulations (dots) versus a fit (dotted line). The fit indicates that $\beta = 0.23$. 

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 210.73.5.33 On: Mon, 28 Apr 2014 16:10:24
By assuming the form (9), we explored numerically what the parameters $\gamma$ and $\beta$ are for different values of the inertial range spectral index $s$. We performed the simulations for pure slab model and a slab/2D composite model.

In Table I, we summarize our findings and we compare our results obtained for slab turbulence with the predication made by SOQLT. As shown, our simulations for slab turbulence agree very well with SOQLT. We also found that for two-component turbulence the values of $\gamma$ and $\beta$ are different compared to the slab simulations. It seems that the chosen turbulence model has an influence on pitch-angle scattering at $90^\circ$. Our results complement the previous simulations performed in Qin and Shalchi. In the latter paper, the pitch-angle Fokker-Planck coefficient was already obtained for two-component turbulence the values of $\gamma$ and $\beta$ are different compared to the slab simulations. It seems that the chosen turbulence model has an influence on pitch-angle scattering at $90^\circ$. Our results complement the previous simulations performed in Qin and Shalchi.

ACKNOWLEDGMENTS

Gang Qin acknowledges support by NNSFC 41125016, NNSFC 4134177, NNSFC 41304135, CMA Grant GYHY201106011, and the Specialized Research Fund for State Key Laboratories of China. The computations were performed by Numerical Forecast Modeling R&D and VR System of State Key Laboratory of Space Weather and Special HPC work stand of Chinese Meridian Project. Andreas Shalchi acknowledges support by the Natural Sciences and Engineering Research Council (NSERC) of Canada.